Strong chains of subsets of ω_1 of length ω_3

Curial Gallart

University of East Anglia

Joint work with David Asperó

Winter School in Abstract Analysis

Hejnice, January 27 – February 3, 2024

Strong chains

Strong chains of functions

Given an ordinal δ, a **strong chain of functions from $\omega_1^{\omega_1}$ of length δ** is a sequence $(f_\alpha : \alpha < \delta)$ of functions $f_\alpha : \omega_1 \to \omega_1$ such that for all $\alpha < \beta < \delta$, $\{\nu \in \omega_1 : f_\alpha(\nu) \geq f_\beta(\nu)\}$ is finite.

Strong chains of functions

Given an ordinal δ, a **strong chain of functions from $\omega_1^{\omega_1}$ of length δ** is a sequence $(f_\alpha : \alpha < \delta)$ of functions $f_\alpha : \omega_1 \to \omega_1$ such that for all $\alpha < \beta < \delta$, $\{\nu \in \omega_1 : f_\alpha(\nu) \geq f_\beta(\nu)\}$ is finite.

Question (Hajnal)

Consistently, are there strong chains of functions from $\omega_1^{\omega_1}$ of length ω_2?

Strong chains of functions

Given an ordinal δ, a **strong chain of functions from $\omega_1^{\omega_1}$ of length δ** is a sequence $(f_\alpha : \alpha < \delta)$ of functions $f_\alpha : \omega_1 \to \omega_1$ such that for all $\alpha < \beta < \delta$, $\{\nu \in \omega_1 : f_\alpha(\nu) \geq f_\beta(\nu)\}$ is finite.

Question (Hajnal)

Consistently, are there strong chains of functions from $\omega_1^{\omega_1}$ of length ω_2?

Answer

- Koszmider (2000). Using forcing with side conditions in morasses.
- Veličković-Venturi (2013). Forcing with Neeman's two-type side conditions.

Strong chains of subsets

Given an ordinal δ, a **strong chain of subsets of ω_1 of length δ** is a sequence $(A_\alpha : \alpha < \delta)$ of subsets of ω_1 such that for all $\alpha < \beta < \delta$,

(1) $|A_\alpha \setminus A_\beta| < \aleph_0$, and

(2) $|A_\beta \setminus A_\alpha| = \aleph_1$.

Strong chains of subsets

Given an ordinal δ, a **strong chain of subsets of ω_1 of length δ** is a sequence $(A_\alpha : \alpha < \delta)$ of subsets of ω_1 such that for all $\alpha < \beta < \delta$,

(1) $|A_\alpha \setminus A_\beta| < \aleph_0$, and

(2) $|A_\beta \setminus A_\alpha| = \aleph_1$.

Question (Hajnal)

Consistently, are there strong chains of subsets of ω_1 of length ω_2?

Strong chains of subsets

Given an ordinal δ, a **strong chain of subsets of ω_1 of length δ** is a sequence $(A_\alpha : \alpha < \delta)$ of subsets of ω_1 such that for all $\alpha < \beta < \delta$,

(1) $|A_\alpha \setminus A_\beta| < \aleph_0$, and

(2) $|A_\beta \setminus A_\alpha| = \aleph_1$.

Question (Hajnal)

Consistently, are there strong chains of subsets of ω_1 of length ω_2?

Equivalently (by identifying each A_α with its characteristic function), a strong chain of subsets of ω_1 of length δ is a sequence $(g_\alpha : \alpha < \delta)$ of functions $g_\alpha : \omega_1 \to 2$ such that for all $\alpha < \beta < \delta$,

(1) $|\{\nu \in \omega_1 : g_\alpha(\nu) > g_\beta(\nu)\}| < \aleph_0$, and

(2) $|\{\nu \in \omega_1 : g_\alpha(\nu) < g_\beta(\nu)\}| = \aleph_1$.

On the length of strong chains

Question

Consistently, are there strong chains of functions from $\omega_1^{\omega_1}$ of length $> \omega_2$?

On the length of strong chains

Question

Consistently, are there strong chains of functions from $\omega_1^{\omega_1}$ of length $> \omega_2$?

Test problem for finding side conditions of models of three types.

On the length of strong chains

Question

Consistently, are there strong chains of functions from $\omega_1^{\omega_1}$ of length $> \omega_2$?

Test problem for finding side conditions of models of three types.

No need for models of three types, two types are enough. Partial answer:

On the length of strong chains

Question

Consistently, are there strong chains of functions from $\omega_1^{\omega_1}$ of length $> \omega_2$?

Test problem for finding side conditions of models of three types.

No need for models of three types, two types are enough. Partial answer:

Theorem (Asperó-G.)

(GCH) There is a forcing notion $\mathbb{P}$ with the following properties:

- $\mathbb{P}$ is proper, $\aleph_1$-proper and has the $\aleph_3$-chain condition.
- $\mathbb{P}$ forces the existence of a strong chain of subsets of ω_1 of length ω_3.

Forcing with side conditions

Review of proper forcing

Let $\mathbb{P}$ be a forcing notion such that $\mathbb{P} \in H(\theta)$ and let $\mathcal{C}$ be a class of elementary submodels of $H(\theta)$.

Review of proper forcing

Let $\mathbb{P}$ be a forcing notion such that $\mathbb{P} \in H(\theta)$ and let $\mathcal{C}$ be a class of elementary submodels of $H(\theta)$.

Definition

- For $M \preceq H(\theta)$ with $\mathbb{P} \in M$, a condition $p \in \mathbb{P}$ is $(M, \mathbb{P})$-**generic** if for every dense $D \subseteq \mathbb{P}$ such that $D \in M$, $D \cap M$ is predense below p.

Review of proper forcing

Let $\mathbb{P}$ be a forcing notion such that $\mathbb{P} \in H(\theta)$ and let $\mathcal{C}$ be a class of elementary submodels of $H(\theta)$.

Definition

- For $M \preceq H(\theta)$ with $\mathbb{P} \in M$, a condition $p \in \mathbb{P}$ is $(M, \mathbb{P})$-**generic** if for every dense $D \subseteq \mathbb{P}$ such that $D \in M$, $D \cap M$ is predense below p.
- $\mathbb{P}$ is $\mathcal{C}$-**proper** if for every $M \in \mathcal{C}$ and every $q \in \mathbb{P} \cap M$ there is $p \leq q$ which is $(M, \mathbb{P})$-generic.

Review of proper forcing

Let $\mathbb{P}$ be a forcing notion such that $\mathbb{P} \in H(\theta)$ and let $\mathcal{C}$ be a class of elementary submodels of $H(\theta)$.

Definition

- For $M \preceq H(\theta)$ with $\mathbb{P} \in M$, a condition $p \in \mathbb{P}$ is $(M, \mathbb{P})$-**generic** if for every dense $D \subseteq \mathbb{P}$ such that $D \in M$, $D \cap M$ is predense below p.
- $\mathbb{P}$ is $\mathcal{C}$-**proper** if for every $M \in \mathcal{C}$ and every $q \in \mathbb{P} \cap M$ there is $p \leq q$ which is $(M, \mathbb{P})$-generic.

Lemma

Let μ be a cardinal. If $\mathbb{P}$ is $\mathcal{C}$-proper and for each $\alpha < \mu$ the set $\{M \in \mathcal{C} : \alpha \subseteq M, |M| < \mu\}$ is stationary in $H(\theta)$, then $\mathbb{P}$ preserves μ.

Forcing with side conditions

Suppose that we want to force an uncountable set with some specific properties. The natural thing would be to force with the poset of all finite approximations. But this poset might collapse cardinals.

Forcing with side conditions

Suppose that we want to force an uncountable set with some specific properties. The natural thing would be to force with the poset of all finite approximations. But this poset might collapse cardinals.

The method of forcing with side conditions, invented by Todorčević, consists on adding elementary submodels to the conditions of a forcing notion $\mathbb{P}$ to ensure the preservation of cardinals.

Forcing with side conditions

Suppose that we want to force an uncountable set with some specific properties. The natural thing would be to force with the poset of all finite approximations. But this poset might collapse cardinals.

The method of forcing with side conditions, invented by Todorčević, consists on adding elementary submodels to the conditions of a forcing notion $\mathbb{P}$ to ensure the preservation of cardinals.

Tipically, a condition of a forcing with side conditions is a pair (x, Δ), where:

Forcing with side conditions

Suppose that we want to force an uncountable set with some specific properties. The natural thing would be to force with the poset of all finite approximations. But this poset might collapse cardinals.

The method of forcing with side conditions, invented by Todorčević, consists on adding elementary submodels to the conditions of a forcing notion $\mathbb{P}$ to ensure the preservation of cardinals.

Tipically, a condition of a forcing with side conditions is a pair (x, Δ), where:

- x, the **working part**, is an approximation of the object that we want to add generically.

Forcing with side conditions

Suppose that we want to force an uncountable set with some specific properties. The natural thing would be to force with the poset of all finite approximations. But this poset might collapse cardinals.

The method of forcing with side conditions, invented by Todorčević, consists on adding elementary submodels to the conditions of a forcing notion $\mathbb{P}$ to ensure the preservation of cardinals.

Tipically, a condition of a forcing with side conditions is a pair (x, Δ), where:

- x, the **working part**, is an approximation of the object that we want to add generically.
- Δ, the **side condition**, is a finite set of elementary submodels of $H(\theta)$.

Forcing with side conditions

Suppose that we want to force an uncountable set with some specific properties. The natural thing would be to force with the poset of all finite approximations. But this poset might collapse cardinals.

The method of forcing with side conditions, invented by Todorčević, consists on adding elementary submodels to the conditions of a forcing notion $\mathbb{P}$ to ensure the preservation of cardinals.

Tipically, a condition of a forcing with side conditions is a pair (x, Δ), where:

- x, the **working part**, is an approximation of the object that we want to add generically.
- Δ, the **side condition**, is a finite set of elementary submodels of $H(\theta)$.
- x and Δ are related in such a way that we can prove that x is $(M, \mathbb{P})$-generic for every $M \in \Delta$.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

(iv) Let $D \subseteq \mathbb{P}$ dense and $D \in M$. Want to show $D \cap M$ is predense below (y, Δ_1).

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

(iv) Let $D \subseteq \mathbb{P}$ dense and $D \in M$. Want to show $D \cap M$ is predense below (y, Δ_1).

(v) Fix $(z, \Delta_2) \leq (y, \Delta_1)$. WLOG $(z, \Delta_2) \in D$.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

(iv) Let $D \subseteq \mathbb{P}$ dense and $D \in M$. Want to show $D \cap M$ is predense below (y, Δ_1).

(v) Fix $(z, \Delta_2) \leq (y, \Delta_1)$. WLOG $(z, \Delta_2) \in D$.

(vi) (Restriction) Tipically, $\exists (z \restriction M, \Delta_2 \restriction M) \in \mathbb{P} \cap M$ such that $(z, \Delta_2) \leq (z \restriction M, \Delta_2 \restriction M)$.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

(iv) Let $D \subseteq \mathbb{P}$ dense and $D \in M$. Want to show $D \cap M$ is predense below (y, Δ_1).

(v) Fix $(z, \Delta_2) \leq (y, \Delta_1)$. WLOG $(z, \Delta_2) \in D$.

(vi) (Restriction) Tipically, $\exists (z \restriction M, \Delta_2 \restriction M) \in \mathbb{P} \cap M$ such that $(z, \Delta_2) \leq (z \restriction M, \Delta_2 \restriction M)$.

(vii) Let φ be a statement about (z, Δ_2) with parameters in M.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

(iv) Let $D \subseteq \mathbb{P}$ dense and $D \in M$. Want to show $D \cap M$ is predense below (y, Δ_1).

(v) Fix $(z, \Delta_2) \leq (y, \Delta_1)$. WLOG $(z, \Delta_2) \in D$.

(vi) (Restriction) Tipically, $\exists (z \restriction M, \Delta_2 \restriction M) \in \mathbb{P} \cap M$ such that $(z, \Delta_2) \leq (z \restriction M, \Delta_2 \restriction M)$.

(vii) Let φ be a statement about (z, Δ_2) with parameters in M.

(viii) $H(\theta) \models$ "$\exists (a, \Sigma) \in \mathbb{P} \cap D$ such that $(a, \Sigma) \leq (z \restriction M, \Delta_2 \restriction M)$ and $\varphi((a, \Sigma))$". Witnessed by (z, Δ_2).

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

(iv) Let $D \subseteq \mathbb{P}$ dense and $D \in M$. Want to show $D \cap M$ is predense below (y, Δ_1).

(v) Fix $(z, \Delta_2) \leq (y, \Delta_1)$. WLOG $(z, \Delta_2) \in D$.

(vi) (Restriction) Tipically, $\exists (z \restriction M, \Delta_2 \restriction M) \in \mathbb{P} \cap M$ such that $(z, \Delta_2) \leq (z \restriction M, \Delta_2 \restriction M)$.

(vii) Let φ be a statement about (z, Δ_2) with parameters in M.

(viii) $H(\theta) \models$ "$\exists (a, \Sigma) \in \mathbb{P} \cap D$ such that $(a, \Sigma) \leq (z \restriction M, \Delta_2 \restriction M)$ and $\varphi((a, \Sigma))$". Witnessed by (z, Δ_2).

(ix) By $M \preceq H(\theta)$, we can find (a, Σ) in M.

Prototypical proof of properness with side conditions

Suppose $\mathbb{P}$ is some forcing with side conditions that we want to show is $\mathcal{C}$-proper.

(i) Let $M \in \mathcal{C}$ and $(x, \Delta_0) \in \mathbb{P} \cap M$.

(ii) (Model on top) Tipically, $\exists (y, \Delta_1) \leq (x, \Delta_0)$ such that $M \in \Delta_1$.

(iii) We want to show that (y, Δ_1) is $(M, \mathbb{P})$-generic.

(iv) Let $D \subseteq \mathbb{P}$ dense and $D \in M$. Want to show $D \cap M$ is predense below (y, Δ_1).

(v) Fix $(z, \Delta_2) \leq (y, \Delta_1)$. WLOG $(z, \Delta_2) \in D$.

(vi) (Restriction) Tipically, $\exists (z \restriction M, \Delta_2 \restriction M) \in \mathbb{P} \cap M$ such that $(z, \Delta_2) \leq (z \restriction M, \Delta_2 \restriction M)$.

(vii) Let φ be a statement about (z, Δ_2) with parameters in M.

(viii) $H(\theta) \models$ "$\exists (a, \Sigma) \in \mathbb{P} \cap D$ such that $(a, \Sigma) \leq (z \restriction M, \Delta_2 \restriction M)$ and $\varphi((a, \Sigma))$". Witnessed by (z, Δ_2).

(ix) By $M \preceq H(\theta)$, we can find (a, Σ) in M.

(x) (Amalgamation) The right φ makes it "easy" to amalgamate (z, Δ_2) and (a, Σ).

The different forms of side conditions

There are two main lineages of side conditions:

The different forms of side conditions

There are two main lineages of side conditions:

- **Linear systems.** Finite sets of models $\{M_i : i \leq n\}$ such that $M_i \in M_{i+1}$, for all $i < n$.

The different forms of side conditions

There are two main lineages of side conditions:

- **Linear systems.** Finite sets of models $\{M_i : i \leq n\}$ such that $M_i \in M_{i+1}$, for all $i < n$.

  The ones initially considered by Todorčević (preservation of $\aleph_1$) and later extended by Neeman to include models of two different cardinalities (preservation of $\aleph_1$ and $\kappa > \aleph_1$, tipically $\aleph_2$).

The different forms of side conditions

There are two main lineages of side conditions:

- **Linear systems.** Finite sets of models $\{M_i : i \leq n\}$ such that $M_i \in M_{i+1}$, for all $i < n$.

  The ones initially considered by Todorčević (preservation of $\aleph_1$) and later extended by Neeman to include models of two different cardinalities (preservation of $\aleph_1$ and $\kappa > \aleph_1$, tipically $\aleph_2$).

- **Symmetric systems.** Finite sets of models not linearly ordered by $\in$, but they exhibit some form of symmetry. Typically, models of the same rank are isomorphic and the system is closed under these isomorphisms.

The different forms of side conditions

There are two main lineages of side conditions:

- **Linear systems.** Finite sets of models $\{M_i : i \leq n\}$ such that $M_i \in M_{i+1}$, for all $i < n$.

  The ones initially considered by Todorčević (preservation of $\aleph_1$) and later extended by Neeman to include models of two different cardinalities (preservation of $\aleph_1$ and $\kappa > \aleph_1$, tipically $\aleph_2$).

- **Symmetric systems.** Finite sets of models not linearly ordered by $\in$, but they exhibit some form of symmetry. Typically, models of the same rank are isomorphic and the system is closed under these isomorphisms.

  Initially considered by Todorčević and later extended by Asperó and Mota (preservation of all cardinals and CH).

Some notation

Assume GCH. Fix a regular cardinal $\kappa > \omega_2$ and let $Q \preceq H(\kappa)$. We denote:

Some notation

Assume GCH. Fix a regular cardinal $\kappa > \omega_2$ and let $Q \preceq H(\kappa)$. We denote:

- $Q \cap \omega_1$ by δ_Q, and call it the ω_1-**height of** Q, and

Some notation

Assume GCH. Fix a regular cardinal $\kappa > \omega_2$ and let $Q \preceq H(\kappa)$. We denote:

- $Q \cap \omega_1$ by δ_Q, and call it the ω_1-**height of** Q, and
- $\sup(Q \cap \omega_2)$ by ε_Q, and call it the ω_2-**height of** Q.

Some notation

Assume GCH. Fix a regular cardinal $\kappa > \omega_2$ and let $Q \preceq H(\kappa)$. We denote:

- $Q \cap \omega_1$ by δ_Q, and call it the ω_1-**height of** Q, and
- $\sup(Q \cap \omega_2)$ by ε_Q, and call it the ω_2-**height of** Q.
- The Skolem hull of $Q \cup \omega_1$ by $Q[\omega_1]$, and call it the ω_1-**hull of** Q. That is, $Q[\omega_1]$ is the least model such that $Q \cup \omega_1 \subseteq Q[\omega_1] \preceq H(\kappa)$.

Some notation

Assume GCH. Fix a regular cardinal $\kappa > \omega_2$ and let $Q \preceq H(\kappa)$. We denote:

- $Q \cap \omega_1$ by δ_Q, and call it the ω_1-**height of** Q, and
- $\sup(Q \cap \omega_2)$ by ε_Q, and call it the ω_2-**height of** Q.
- The Skolem hull of $Q \cup \omega_1$ by $Q[\omega_1]$, and call it the ω_1-**hull of** Q. That is, $Q[\omega_1]$ is the least model such that $Q \cup \omega_1 \subseteq Q[\omega_1] \preceq H(\kappa)$.

If $Q_0, Q_1 \preceq H(\kappa)$ and $(Q_0[\omega_1], \in, Q_0) \cong (Q_1[\omega_1], \in, Q_1)$, we denote by $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ the unique isomorphism between these structures.

Some notation

Assume GCH. Fix a regular cardinal $\kappa > \omega_2$ and let $Q \preceq H(\kappa)$. We denote:

- $Q \cap \omega_1$ by δ_Q, and call it the ω_1-**height of** Q, and
- $\sup(Q \cap \omega_2)$ by ε_Q, and call it the ω_2-**height of** Q.
- The Skolem hull of $Q \cup \omega_1$ by $Q[\omega_1]$, and call it the ω_1-**hull of** Q. That is, $Q[\omega_1]$ is the least model such that $Q \cup \omega_1 \subseteq Q[\omega_1] \preceq H(\kappa)$.

If $Q_0, Q_1 \preceq H(\kappa)$ and $(Q_0[\omega_1], \in, Q_0) \cong (Q_1[\omega_1], \in, Q_1)$, we denote by $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ the unique isomorphism between these structures.

Let $P \subseteq H(\kappa)$. Let $\mathcal{S}$ be the set of countable $M \preceq (H(\kappa); \in, P)$ and $\mathcal{L}$ be the set of $N \preceq (H(\kappa); \in, P)$ such that $|N| = \aleph_1$ and ${}^{\omega}N \subseteq N$.

Two-type symmetric systems

Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-**symmetric system** if and only if the following holds:

Two-type symmetric systems

Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-**symmetric system** if and only if the following holds:

(A) Every $Q \in \mathcal{N}$ is an element of $\mathcal{S} \cup \mathcal{L}$.

Two-type symmetric systems

Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-**symmetric system** if and only if the following holds:

(A) Every $Q \in \mathcal{N}$ is an element of $\mathcal{S} \cup \mathcal{L}$.

(B) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, then $(Q_0[\omega_1], \in, Q_0) \cong (Q_1[\omega_1], \in, Q_1)$, and $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

Two-type symmetric systems

Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-**symmetric system** if and only if the following holds:

(A) Every $Q \in \mathcal{N}$ is an element of $\mathcal{S} \cup \mathcal{L}$.

(B) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, then $(Q_0[\omega_1], \in, Q_0) \cong (Q_1[\omega_1], \in, Q_1)$, and $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

(C) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, then there is $Q_1' \in \mathcal{N}$ such that $\varepsilon_{Q_1'} = \varepsilon_{Q_1}$ and $Q_0 \in Q_1'[\omega_1]$.

Two-type symmetric systems

Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-**symmetric system** if and only if the following holds:

(A) Every $Q \in \mathcal{N}$ is an element of $\mathcal{S} \cup \mathcal{L}$.

(B) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, then $(Q_0[\omega_1], \in, Q_0) \cong (Q_1[\omega_1], \in, Q_1)$, and $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

(C) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, then there is $Q_1' \in \mathcal{N}$ such that $\varepsilon_{Q_1'} = \varepsilon_{Q_1}$ and $Q_0 \in Q_1'[\omega_1]$.

(D) For every $Q \in \mathcal{N}$ and every $M \in \mathcal{N} \cap \mathcal{S}$, if $Q \in M[\omega_1]$ and there is no $Q' \in \mathcal{N}$ such that $\varepsilon_Q < \varepsilon_{Q'} < \varepsilon_M$, then in fact $Q \in M$.

Two-type symmetric systems

Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-**symmetric system** if and only if the following holds:

(A) Every $Q \in \mathcal{N}$ is an element of $\mathcal{S} \cup \mathcal{L}$.

(B) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, then $(Q_0[\omega_1], \in, Q_0) \cong (Q_1[\omega_1], \in, Q_1)$, and $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

(C) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, then there is $Q_1' \in \mathcal{N}$ such that $\varepsilon_{Q_1'} = \varepsilon_{Q_1}$ and $Q_0 \in Q_1'[\omega_1]$.

(D) For every $Q \in \mathcal{N}$ and every $M \in \mathcal{N} \cap \mathcal{S}$, if $Q \in M[\omega_1]$ and there is no $Q' \in \mathcal{N}$ such that $\varepsilon_Q < \varepsilon_{Q'} < \varepsilon_M$, then in fact $Q \in M$.

(E) For every $N \in \mathcal{N} \cap \mathcal{L}$ and every $M \in \mathcal{N} \cap \mathcal{S}$, if $N \in M$, then $N \cap M \in \mathcal{N}$.

Two-type symmetric systems

Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-**symmetric system** if and only if the following holds:

(A) Every $Q \in \mathcal{N}$ is an element of $\mathcal{S} \cup \mathcal{L}$.

(B) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, then $(Q_0[\omega_1], \in, Q_0) \cong (Q_1[\omega_1], \in, Q_1)$, and $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

(C) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, then there is $Q_1' \in \mathcal{N}$ such that $\varepsilon_{Q_1'} = \varepsilon_{Q_1}$ and $Q_0 \in Q_1'[\omega_1]$.

(D) For every $Q \in \mathcal{N}$ and every $M \in \mathcal{N} \cap \mathcal{S}$, if $Q \in M[\omega_1]$ and there is no $Q' \in \mathcal{N}$ such that $\varepsilon_Q < \varepsilon_{Q'} < \varepsilon_M$, then in fact $Q \in M$.

(E) For every $N \in \mathcal{N} \cap \mathcal{L}$ and every $M \in \mathcal{N} \cap \mathcal{S}$, if $N \in M$, then $N \cap M \in \mathcal{N}$.

(F) For all $Q_0, Q_1, Q_1' \in \mathcal{N}$ such that $Q_0 \in Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$, $\Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_0) \in \mathcal{N}$.

Pure side condition forcing

Let $\mathbb{M}$ be the poset of all $(\mathcal{S}, \mathcal{L})$-symmetric systems ordered by reverse inclusion (**pure side condition forcing**).

Pure side condition forcing

Let $\mathbb{M}$ be the poset of all $(\mathcal{S}, \mathcal{L})$-symmetric systems ordered by reverse inclusion (**pure side condition forcing**).

$\mathbb{M}$ has the three properties required for a forcing with side conditions to get properness: "Model on top", "restriction", and "amalgamation".

Pure side condition forcing

Let $\mathbb{M}$ be the poset of all $(\mathcal{S}, \mathcal{L})$-symmetric systems ordered by reverse inclusion (**pure side condition forcing**).

$\mathbb{M}$ has the three properties required for a forcing with side conditions to get properness: "Model on top", "restriction", and "amalgamation".

Lemma

- $\mathbb{M}$ is $\mathcal{S}$-(strongly) proper, and thus it preserves $\aleph_1$.

Pure side condition forcing

Let $\mathbb{M}$ be the poset of all $(\mathcal{S}, \mathcal{L})$-symmetric systems ordered by reverse inclusion (**pure side condition forcing**).

$\mathbb{M}$ has the three properties required for a forcing with side conditions to get properness: "Model on top", "restriction", and "amalgamation".

Lemma

- $\mathbb{M}$ is $\mathcal{S}$-(strongly) proper, and thus it preserves $\aleph_1$.
- (CH) $\mathbb{M}$ is $\mathcal{L}$-(strongly) proper, and thus it preserves $\aleph_2$.

Pure side condition forcing

Let $\mathbb{M}$ be the poset of all $(\mathcal{S}, \mathcal{L})$-symmetric systems ordered by reverse inclusion (**pure side condition forcing**).

$\mathbb{M}$ has the three properties required for a forcing with side conditions to get properness: "Model on top", "restriction", and "amalgamation".

Lemma

- $\mathbb{M}$ is $\mathcal{S}$-(strongly) proper, and thus it preserves $\aleph_1$.
- (CH) $\mathbb{M}$ is $\mathcal{L}$-(strongly) proper, and thus it preserves $\aleph_2$.
- $(2^{\aleph_1} = \aleph_2)$ $\mathbb{M}$ has the $\aleph_3$-c.c., and thus it preserves all cardinals $\geq \aleph_3$.

Pure side condition forcing

Let $\mathbb{M}$ be the poset of all $(\mathcal{S}, \mathcal{L})$-symmetric systems ordered by reverse inclusion (**pure side condition forcing**).

$\mathbb{M}$ has the three properties required for a forcing with side conditions to get properness: "Model on top", "restriction", and "amalgamation".

Lemma

- $\mathbb{M}$ is $\mathcal{S}$-(strongly) proper, and thus it preserves $\aleph_1$.
- (CH) $\mathbb{M}$ is $\mathcal{L}$-(strongly) proper, and thus it preserves $\aleph_2$.
- $(2^{\aleph_1} = \aleph_2)$ $\mathbb{M}$ has the $\aleph_3$-c.c., and thus it preserves all cardinals $\geq \aleph_3$.
- $(2^{\aleph_1} = \aleph_2)$ $\mathbb{M}$ preserves $2^{\aleph_1} = \aleph_2$.

Forcing strong chains

First attempt at forcing strong chains

Recall that we want to force a sequence $(g_\alpha : \alpha < \omega_3)$ of functions $g_\alpha : \omega_1 \to 2$ such that for all $\alpha < \beta < \omega_3$,

(1) $|\{\nu \in \omega_1 : g_\alpha(\nu) > g_\beta(\nu)\}| < \aleph_0$, and

(2) $|\{\nu \in \omega_1 : g_\alpha(\nu) < g_\beta(\nu)\}| = \aleph_1$.

First attempt at forcing strong chains

Recall that we want to force a sequence $(g_\alpha : \alpha < \omega_3)$ of functions $g_\alpha : \omega_1 \to 2$ such that for all $\alpha < \beta < \omega_3$,

(1) $|\{\nu \in \omega_1 : g_\alpha(\nu) > g_\beta(\nu)\}| < \aleph_0$, and

(2) $|\{\nu \in \omega_1 : g_\alpha(\nu) < g_\beta(\nu)\}| = \aleph_1$.

The first impulse would be to use conditions of the form $p = (x_p, \Delta_p)$ such that:

First attempt at forcing strong chains

Recall that we want to force a sequence $(g_\alpha : \alpha < \omega_3)$ of functions $g_\alpha : \omega_1 \to 2$ such that for all $\alpha < \beta < \omega_3$,

(1) $|\{\nu \in \omega_1 : g_\alpha(\nu) > g_\beta(\nu)\}| < \aleph_0$, and

(2) $|\{\nu \in \omega_1 : g_\alpha(\nu) < g_\beta(\nu)\}| = \aleph_1$.

The first impulse would be to use conditions of the form $p = (x_p, \Delta_p)$ such that:

- $a_p \in [\omega_3]^{<\omega}$, $d_p \in [\omega_1]^{<\omega}$, and $x_p^\alpha : d_p \to 2$ is an approximation of g_α, for each $\alpha \in a_p$.

First attempt at forcing strong chains

Recall that we want to force a sequence $(g_\alpha : \alpha < \omega_3)$ of functions $g_\alpha : \omega_1 \to 2$ such that for all $\alpha < \beta < \omega_3$,

(1) $|\{\nu \in \omega_1 : g_\alpha(\nu) > g_\beta(\nu)\}| < \aleph_0$, and

(2) $|\{\nu \in \omega_1 : g_\alpha(\nu) < g_\beta(\nu)\}| = \aleph_1$.

The first impulse would be to use conditions of the form $p = (x_p, \Delta_p)$ such that:

- $a_p \in [\omega_3]^{<\omega}$, $d_p \in [\omega_1]^{<\omega}$, and $x_p^\alpha : d_p \to 2$ is an approximation of g_α, for each $\alpha \in a_p$.
- Δ_p is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

First attempt at forcing strong chains

Recall that we want to force a sequence $(g_\alpha : \alpha < \omega_3)$ of functions $g_\alpha : \omega_1 \to 2$ such that for all $\alpha < \beta < \omega_3$,

(1) $|\{\nu \in \omega_1 : g_\alpha(\nu) > g_\beta(\nu)\}| < \aleph_0$, and

(2) $|\{\nu \in \omega_1 : g_\alpha(\nu) < g_\beta(\nu)\}| = \aleph_1$.

The first impulse would be to use conditions of the form $p = (x_p, \Delta_p)$ such that:

- $a_p \in [\omega_3]^{<\omega}$, $d_p \in [\omega_1]^{<\omega}$, and $x_p^\alpha : d_p \to 2$ is an approximation of g_α, for each $\alpha \in a_p$.
- Δ_p is an $(\mathcal{S}, \mathcal{L})$-symmetric system.
- If $\alpha < \beta$ are in $M \cap \omega_3$ for some $M \in \Delta_p \cap \mathcal{S}$, then M should localize the disagreement of x_p^α and x_p^β, i.e., p should force that the finite set $\{\nu < \omega_1 : x_p^\alpha(\nu) > x_p^\beta(\nu)\}$ belongs to M.

First attempt at forcing strong chains

Recall that we want to force a sequence $(g_\alpha : \alpha < \omega_3)$ of functions $g_\alpha : \omega_1 \to 2$ such that for all $\alpha < \beta < \omega_3$,

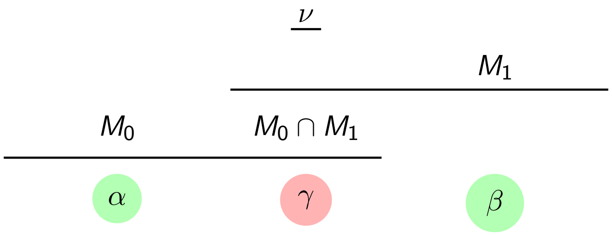
(1) $|\{\nu \in \omega_1 : g_\alpha(\nu) > g_\beta(\nu)\}| < \aleph_0$, and

(2) $|\{\nu \in \omega_1 : g_\alpha(\nu) < g_\beta(\nu)\}| = \aleph_1$.

The first impulse would be to use conditions of the form $p = (x_p, \Delta_p)$ such that:

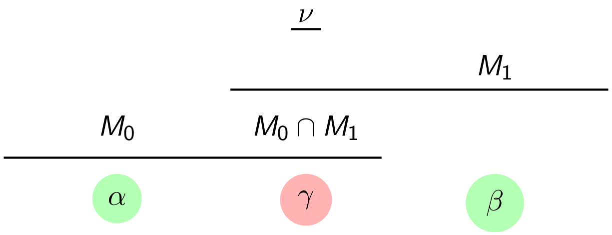
- $a_p \in [\omega_3]^{<\omega}$, $d_p \in [\omega_1]^{<\omega}$, and $x_p^\alpha : d_p \to 2$ is an approximation of g_α, for each $\alpha \in a_p$.
- Δ_p is an $(\mathcal{S}, \mathcal{L})$-symmetric system.
- If $\alpha < \beta$ are in $M \cap \omega_3$ for some $M \in \Delta_p \cap \mathcal{S}$, then M should localize the disagreement of x_p^α and x_p^β, i.e., p should force that the finite set $\{\nu < \omega_1 : x_p^\alpha(\nu) > x_p^\beta(\nu)\}$ belongs to M.
- Equivalently, if $\alpha, \beta \in M$ and $\nu \in d_p \setminus M$, then $x_p^\alpha(\nu) \leq x_p^\beta(\nu)$.

This forcing won't work!



Suppose $\alpha < \beta$ in a_p, and $\nu \in d_p \setminus (M_0 \cup M_1)$. We could have $x_p^{\alpha}(\nu) > x_p^{\beta}(\nu)$. Suppose that $q \leq p$ and $\gamma \in a_q \setminus a_p \cap (\alpha, \beta) \cap M_0 \cap M_1$. Then, $x_q^{\alpha}(\nu) \leq x_q^{\gamma}(\nu) \leq x_q^{\beta}(\nu)$.

This forcing won't work!



Suppose $\alpha < \beta$ in a_p, and $\nu \in d_p \setminus (M_0 \cup M_1)$. We could have $x_p^\alpha(\nu) > x_p^\beta(\nu)$. Suppose that $q \le p$ and $\gamma \in a_q \setminus a_p \cap (\alpha, \beta) \cap M_0 \cap M_1$. Then, $x_q^\alpha(\nu) \le x_q^\gamma(\nu) \le x_q^\beta(\nu)$.

Definition

Let $\mathcal{A}$ be a finite subset of $\mathcal{S}$, $\nu \in \omega_1$, and $\alpha, \beta \in \omega_3$. Then, $\alpha <_{\mathcal{A},\nu} \beta$ if and only if $\alpha < \beta$ and there are $M_0, \ldots, M_n \in \mathcal{A}$ and $\gamma_0 < \cdots < \gamma_{n-1}$ such that $\sup_{i \le n} \delta_{M_i} \le \nu$, $\alpha \in M_0$, $\beta \in M_n$, and $\gamma_i \in M_i \cap M_{i+1} \cap (\alpha, \beta)$ for each $i < n$.

The forcing

Let $\mathbb{P}$ be the forcing whose conditions are tuples $p = (x_p, a_p, d_p, \mathcal{N}_p, \mathcal{A}_p)$ such that:

The forcing

Let $\mathbb{P}$ be the forcing whose conditions are tuples $p = (x_p, a_p, d_p, \mathcal{N}_p, \mathcal{A}_p)$ such that:

(1) $a_p \in [\omega_3]^{<\omega}$.

(2) $d_p \in [\omega_1]^{<\omega}$.

(3) $x_p = (x_p^\alpha : \alpha \in a_p)$ and $x_p^\alpha : d_p \to 2$ is a function for each $\alpha \in a_p$.

(4) $\mathcal{N}_p$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

The forcing

Let $\mathbb{P}$ be the forcing whose conditions are tuples $p = (x_p, a_p, d_p, \mathcal{N}_p, \mathcal{A}_p)$ such that:

(1) $a_p \in [\omega_3]^{<\omega}$.

(2) $d_p \in [\omega_1]^{<\omega}$.

(3) $x_p = (x_p^\alpha : \alpha \in a_p)$ and $x_p^\alpha : d_p \to 2$ is a function for each $\alpha \in a_p$.

(4) $\mathcal{N}_p$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

(5) $\mathcal{A}_p \subseteq \mathcal{N}_p \cap \mathcal{S}$ is such that $N \cap M \in \mathcal{A}_p$ for all $M \in \mathcal{A}_p$ and $N \in \mathcal{N}_p \cap \mathcal{L} \cap M$.

The forcing

Let $\mathbb{P}$ be the forcing whose conditions are tuples $p = (x_p, a_p, d_p, \mathcal{N}_p, \mathcal{A}_p)$ such that:

(1) $a_p \in [\omega_3]^{<\omega}$.

(2) $d_p \in [\omega_1]^{<\omega}$.

(3) $x_p = (x_p^\alpha : \alpha \in a_p)$ and $x_p^\alpha : d_p \to 2$ is a function for each $\alpha \in a_p$.

(4) $\mathcal{N}_p$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

(5) $\mathcal{A}_p \subseteq \mathcal{N}_p \cap \mathcal{S}$ is such that $N \cap M \in \mathcal{A}_p$ for all $M \in \mathcal{A}_p$ and $N \in \mathcal{N}_p \cap \mathcal{L} \cap M$.

(6) $\forall \nu \in d_p \forall \alpha, \beta \in a_p$, if $\alpha <_{\mathcal{A}_p, \nu} \beta$, then $x_p^\alpha(\nu) \leq x_p^\beta(\nu)$.

The forcing

Let $\mathbb{P}$ be the forcing whose conditions are tuples $p = (x_p, a_p, d_p, \mathcal{N}_p, \mathcal{A}_p)$ such that:

(1) $a_p \in [\omega_3]^{<\omega}$.

(2) $d_p \in [\omega_1]^{<\omega}$.

(3) $x_p = (x_p^\alpha : \alpha \in a_p)$ and $x_p^\alpha : d_p \to 2$ is a function for each $\alpha \in a_p$.

(4) $\mathcal{N}_p$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

(5) $\mathcal{A}_p \subseteq \mathcal{N}_p \cap \mathcal{S}$ is such that $N \cap M \in \mathcal{A}_p$ for all $M \in \mathcal{A}_p$ and $N \in \mathcal{N}_p \cap \mathcal{L} \cap M$.

(6) $\forall \nu \in d_p \forall \alpha, \beta \in a_p$, if $\alpha <_{\mathcal{A}_{p,\nu}} \beta$, then $x_p^\alpha(\nu) \leq x_p^\beta(\nu)$.

Given $p, q \in \mathbb{P}$, $q \leq p$ if and only if $\mathcal{N}_q \supseteq \mathcal{N}_p$, $\mathcal{A}_q \supseteq \mathcal{A}_p$, $a_q \supseteq a_p$, $d_q \supseteq d_p$, and $x_q^\alpha \supseteq x_p^\alpha$, for all $\alpha \in a_p$.

Basic properties of the forcing

Lemma (model on top)

Let $Q \in \mathcal{S} \cup \mathcal{L}$ and $p \in \mathbb{P} \cap Q$. Then, there is a condition $q \leq p$ such that

- $Q \in \mathcal{N}_q$, and
- $Q \in \mathcal{A}_q$, if $Q \in \mathcal{S}$.

Basic properties of the forcing

Lemma (model on top)

Let $Q \in \mathcal{S} \cup \mathcal{L}$ and $p \in \mathbb{P} \cap Q$. Then, there is a condition $q \leq p$ such that

- $Q \in \mathcal{N}_q$, and
- $Q \in \mathcal{A}_q$, if $Q \in \mathcal{S}$.

Definition

Let $p = (x_p, a_p, d_p, \mathcal{N}_p, \mathcal{A}_p)$ and $Q \in \mathcal{N}_p$. Then, we define $p \restriction Q$ by letting $\mathcal{N}_{p \restriction Q} = \mathcal{N}_p \cap Q$, $\mathcal{A}_{p \restriction Q} = \mathcal{A}_p \cap Q$, $a_{p \restriction Q} = a_p \cap Q$, $d_{p \restriction Q} = d_p \cap Q$, and $x^{\alpha}_{p \restriction Q}(\nu) = x^{\alpha}_p(\nu)$, for all $\alpha \in a_{p \restriction Q}$ and $\nu \in d_{p \restriction Q}$.

Lemma (model on top)

Let $Q \in \mathcal{S} \cup \mathcal{L}$ and $p \in \mathbb{P} \cap Q$. Then, there is a condition $q \leq p$ such that

- $Q \in \mathcal{N}_q$, and
- $Q \in \mathcal{A}_q$, if $Q \in \mathcal{S}$.

Definition

Let $p = (x_p, a_p, d_p, \mathcal{N}_p, \mathcal{A}_p)$ and $Q \in \mathcal{N}_p$. Then, we define $p \restriction Q$ by letting $\mathcal{N}_{p\restriction Q} = \mathcal{N}_p \cap Q$, $\mathcal{A}_{p\restriction Q} = \mathcal{A}_p \cap Q$, $a_{p\restriction Q} = a_p \cap Q$, $d_{p\restriction Q} = d_p \cap Q$, and $x^{\alpha}_{p\restriction Q}(\nu) = x^{\alpha}_p(\nu)$, for all $\alpha \in a_{p\restriction Q}$ and $\nu \in d_{p\restriction Q}$.

Lemma (restriction)

If $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$, then $p \restriction Q \in \mathbb{P} \cap Q$ and $p \leq p \restriction Q$, if $Q \in \mathcal{L}$ or $Q \in \mathcal{A}_p$.

Preservation theorems

Amalgamation lemma

Let $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$. Let $q \in \mathbb{P} \cap Q$ and suppose the following holds:

Amalgamation lemma

Let $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$. Let $q \in \mathbb{P} \cap Q$ and suppose the following holds:

- $q \leq p \restriction Q$.

Preservation theorems

Amalgamation lemma

Let $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$. Let $q \in \mathbb{P} \cap Q$ and suppose the following holds:

- $q \leq p \restriction Q$.
- φ is a carefully chosen first-order formula with parameters in Q such that $H(\omega_3) \models \varphi(p)$, and $Q \models \varphi(q)$.

Preservation theorems

Amalgamation lemma

Let $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$. Let $q \in \mathbb{P} \cap Q$ and suppose the following holds:

- $q \leq p \restriction Q$.
- φ is a carefully chosen first-order formula with parameters in Q such that $H(\omega_3) \models \varphi(p)$, and $Q \models \varphi(q)$.

Then there is $r \in \mathbb{P}$ such that $r \leq p, q$.

Preservation theorems

Amalgamation lemma

Let $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$. Let $q \in \mathbb{P} \cap Q$ and suppose the following holds:

- $q \leq p \restriction Q$.
- φ is a carefully chosen first-order formula with parameters in Q such that $H(\omega_3) \models \varphi(p)$, and $Q \models \varphi(q)$.

Then there is $r \in \mathbb{P}$ such that $r \leq p, q$.

Corollary

$\mathbb{P}$ is $\mathcal{S}$-proper and $\mathcal{L}$-proper. So, $\mathbb{P}$ preserves $\aleph_1$ and $\aleph_2$.

Preservation theorems

Amalgamation lemma

Let $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$. Let $q \in \mathbb{P} \cap Q$ and suppose the following holds:

- $q \leq p \restriction Q$.
- φ is a carefully chosen first-order formula with parameters in Q such that $H(\omega_3) \models \varphi(p)$, and $Q \models \varphi(q)$.

Then there is $r \in \mathbb{P}$ such that $r \leq p, q$.

Corollary

$\mathbb{P}$ is $\mathcal{S}$-proper and $\mathcal{L}$-proper. So, $\mathbb{P}$ preserves $\aleph_1$ and $\aleph_2$.

Lemma

($2^{\aleph_1} = \aleph_2$) $\mathbb{P}$ has the $\aleph_3$-chain condition. So, $\mathbb{P}$ preserves all cardinals $\geq \aleph_3$.

The result

Theorem (Asperó-G.)

(GCH) There is a forcing notion $\mathbb{P}$ with the following properties:

- $\mathbb{P}$ is proper, $\aleph_1$-proper and has the $\aleph_3$-chain condition.
- $\mathbb{P}$ forces the existence of a strong chain of subsets of ω_1 of length ω_3.

Future work

A variation of the forcing should lead to the consistency of the existence of strong chains of length ω_3 of functions from ω_1 to ω_1. Needs a little bit more work.

Future work

A variation of the forcing should lead to the consistency of the existence of strong chains of length ω_3 of functions from ω_1 to ω_1. Needs a little bit more work.

$(\mathcal{S}, \mathcal{L})$-symmetric systems should be useful to get the consistency of other combinatorial statements.

Future work

A variation of the forcing should lead to the consistency of the existence of strong chains of length ω_3 of functions from ω_1 to ω_1. Needs a little bit more work.

$(\mathcal{S}, \mathcal{L})$-symmetric systems should be useful to get the consistency of other combinatorial statements.

Question

Can we get strong chains of functions from $\omega_1^{\omega_1}$ of length $> \omega_3$?

Thank you for your attention!